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# Special Uniform Approximations of Continuous Vector-Valued Functions. Part I: Special Approximations in $C_X(T)$

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In this paper, we give special uniform approximations of functions $u$ from the spaces $C_X(T)$ and $C_\infty(T,X)$, with elements $\bar{u}$ of the tensor products $C_\Gamma(T) \otimes X$, respectively $C_0(T,\Gamma) \otimes X$, for a topological space $T$ and a $\Gamma$-locally convex space $X$. We call an approximation special, if $\bar{u}$ satisfies additional constraints, namely $\operatorname{supp} v \subset u^{-1}(X \setminus \{0\})$ and $\bar{u}(T) \subset \operatorname{co}(u(T))$ (resp. $\subset \operatorname{co}(u(T) \cup \{0\})$). In Section 3, we give three distinct applications, which are due exactly to these constraints: a density result with respect to the inductive limit topology, a Tietze–Dugundji's type extension new theorem and a proof of Schauder–Tihonov's fixed point theorem.

*Key Words:* uniform approximation; tensor product; extension by continuity.

## 1. INTRODUCTION

Throughout this paper, $T$ is a topological space, $X$ a locally convex space over the field $\Gamma \in \{\mathbf{R}, \mathbf{C}\}$ and $C_X(T)$ the linear space of all $X$-valued continuous functions on $T$. Consider the vector subspaces

$$C_{\mathrm{b}}(T,X) := \{u \in C_X(T) \mid u(T) \text{ is bounded}\} \subset C_X(T),$$

$$C_{\mathrm{tb}}(T,X) := \{u \in C_X(T) \mid u(T) \text{ is totally bounded}\} \subset C_{\mathrm{b}}(T,X).$$

Recall that a subset $A \subset X$ is said to be totally bounded iff for every $W \in \mathscr{V}_X(0)$, there exists a finite subset $A_0 \subset X$, such that $A \subset A_0 + W$ (then, we can choose $A_0 \subset A$). If $T$ and $X$ are both Hausdorff spaces, we also use the standard notations

$$C_\infty(T,X) := \{u \in C_X(T) \mid \forall W \in \mathscr{V}_X(0),\ u^{-1}(X \setminus \overset{\circ}{W}) \text{ is compact}\},$$

$$C_0(T,X) := \{u \in C_X(T) \mid \operatorname{supp} u := \overline{u^{-1}(X \setminus \{0\})} \text{ is compact}\}.$$



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It is obvious that $C_0(T,X) \subset C_\infty(T,X) \subset C_{\mathrm{tb}}(T,X) \subset C_{\mathrm{b}}(T,X)$. We have the natural inclusions

$$C_\Gamma(T) \otimes X \subset C_X(T), \qquad C_0(T,\Gamma) \otimes X \subset C_0(T,X).$$

Various results concerning the uniform density of $C_\Gamma(T) \otimes X$ in $C_X(T)$ and Weierstrass–Stone's type theorems are known (see [1, 4–8]). Therefore, we will restrict our attention to special uniform approximations and its applications.

## 2. SPECIAL APPROXIMATIONS IN $C_X(T)$

### *2.1. The vector space* $(C_\Gamma(T) \otimes X)_{\mathrm{loc}}$

It is easily seen that if $E$ is a $\Gamma$-normed space and if $u \in C_{\mathrm{b}}(T,E)$ has the following uniform approximation property:

$$\forall \varepsilon > 0,\ \exists u_\varepsilon \in C_\Gamma(T) \otimes E,\ \text{such that } \|u - u_\varepsilon\|_\infty < \varepsilon,$$

then $u \in C_{\mathrm{tb}}(T,E)$. Therefore, to get $\varepsilon$-uniform approximations of $u$ for arbitrary $\varepsilon > 0$, we have to accept $u \in C_{\mathrm{tb}}(T,E)$ (Theorem 1 will prove that this condition is also sufficient, even for special approximations) or to replace the vector subspace $C_\Gamma(T) \otimes E$ of $C_E(T)$ by a larger one. This is a reason for:

DEFINITION 1. Consider the "locally tensor product"

$$(C_\Gamma(T) \otimes X)_{\mathrm{loc}} := \{u : T \to X \mid \forall t \in T,\ \exists V \in \mathscr{V}_T(t),$$

$$\exists v \in C_\Gamma(T) \otimes X,\ \text{such that } u_{|_V} = v_{|_V}\}.$$

PROPOSITION 1. (1) $(C_\Gamma(T) \otimes X)_{\mathrm{loc}}$ *is a* $\Gamma$*-vector space and*

$$C_\Gamma(T) \otimes X \subset (C_\Gamma(T) \otimes X)_{\mathrm{loc}} \subset C_X(T).$$

(2) *If* $T$ *is compact, then* $(C_\Gamma(T) \otimes X)_{\mathrm{loc}} = C_\Gamma(T) \otimes X$.

*Proof.* Statement (1) is evident. The proof of (2) is immediate, using a partition of unity (p.u.) on $T$. ∎

*Remark* 1. If $u \in (C_\Gamma(T) \otimes X)_{\mathrm{loc}}$ and $K$ is a compact subset of $T$, then $u_{|_K} \in C_\Gamma(K) \otimes X$.

*2.2. The uniform density of* $(C_\Gamma(T) \otimes X)_{\text{loc}}$ *in* $C_X(T)$

THEOREM 1. *Consider* $u \in C_X(T)$. *If* $T$ *or* $u(T)$ *is paracompact or if* $u \in C_{\text{tb}}(T, X)$, *then for every* $W \in \mathscr{V}_X(0)$, *there exists an approximant* $u_W \in (C_\Gamma(T) \otimes X)_{\text{loc}}$, *such that*:

(1) $(u - u_W)(T) \subset W,\ u_W(T) \subset \text{co}(u(T)),\ \text{supp}\, u_W \subset u^{-1}(X \setminus \{0\})$,

(2) $u_W = \sum_{i \in I} \varphi_i(\cdot)x_i$, *with* $(x_i)_{i \in I} \subset u(T)$ *and* $(\varphi_i)_{i \in I}$ *p.u. on* $T$. *Moreover, if* $u \in C_{\text{tb}}(T, X)$, *then* $I$ *can be choosen as a finite set and, consequently,* $u_W \in C_\Gamma(T) \otimes X$.

*Proof.* Fix $W \in \mathscr{V}_X(0)$. We can certainly assume that $W$ is open and convex. If $u \in C_{\text{tb}}(T, X)$, then $\exists A_0 \subset u(T)$, such that $A_0$ is finite and $u(T) \subset A_0 + 2^{-1}W$. Set $A := A_0$ if $u \in C_{\text{tb}}(T, X)$, and $A := u(T)$, otherwise. Thus, $u(T) \subset A + W$, and so $T = \bigcup_{x \in A} u^{-1}(x + W)$. There are three cases to consider:

(a) If $T$ is paracompact, then $\exists (\varphi_x)_{x \in A}$ p.u. on $T$, subordinated to the open covering $(u^{-1}(x + W))_{x \in A}$ of $T$.

(b) If $u(T)$ is paracompact, then $\exists (\psi_x)_{x \in A}$ p.u. on $u(T)$, subordinated to the open covering $((x + W) \cap u(T))_{x \in A}$ of $u(T)$. For $x \in A$, set $\varphi_x := \psi_x \circ u$. Hence, $\text{supp}\, \varphi_x \subset u^{-1}(\text{supp}\, \psi_x) \subset u^{-1}(x + W)\ \forall x \in A$.

(c) If $u \in C_{\text{tb}}(T, X)$, then $A$ is finite and $u(T) \subset A + 2^{-1}W$. Define the map $\omega : X \to [0, 1],\ \omega(z) = 0 \vee [1 - 2p_W(z)]$, where $p_W$ means Minkowski's functional associated to $W$. Clearly, $\omega$ is continuous and $\text{supp}\, \omega \subset 2^{-1}\bar{W}$. For every $x \in A$, define $\omega_x : u(T) \to [0, 1],\ \omega_x(z) = \omega(z - x)$. But $\forall z \in u(T) \subset A + 2^{-1}W,\ \exists x \in A$, such that $z \in x + 2^{-1}W$, which gives $\omega_x(z) = \omega(z - x) > 0$. Since $\sum_{y \in A} \omega_y > 0$ on $u(T)$, we can define the map $\psi_x = (\sum_{y \in A} \omega_y)^{-1}\omega_x : u(T) \to [0, 1],\quad \varphi_x := \psi_x \circ u\ \forall x \in A$. Clearly, $\text{supp}\, \psi_x \subset (x + 2^{-1}\bar{W}) \cap u(T) \subset (x + W) \cap u(T)$, $\text{supp}\, \varphi_x \subset u^{-1}(\text{supp}\, \psi_x) \subset u^{-1}(x + W)\ \forall x \in A$.

In all the above three cases, $(\varphi_x)_{x \in A}$ p.u. on $T$, subordinated to the open covering $(u^{-1}(x + W))_{x \in A}$ of $T$. Now set $v := \sum_{x \in A} \varphi_x(\cdot)x \in (C_\Gamma(T) \otimes X)_{\text{loc}}$. Obviously, $v(T) \subset \text{co}(u(T))$. We next show that $(u - v)(T) \subset W$. Fix $t \in T$ and set $A_t := \{x \in A \mid \varphi_x(t) \neq 0\}$. Thus, $A_t$ is finite, $\sum_{x \in A_t} \varphi_x(t) = 1$ and $\forall x \in A_t$, we have $t \in \text{supp}\, \varphi_x \subset u^{-1}(x + W)$, and so $(u - v)(t) = \sum_{x \in A_t} \varphi_x(t)(u(t) - x) \in \sum_{x \in A_t} \varphi_x(t)W = W$. Therefore, $(u - v)(T) \subset W$. We need consider two cases:

(i) If $0 \notin u(T)$, then $u^{-1}(X \setminus \{0\}) = T \supset \text{supp}\, v$, and so $u_W := v$ satisfies all required properties.

(ii) If $0 \in u(T)$, choose $\varepsilon \in (0, 1)$. Clearly, $\varepsilon p\bar{W} \subset \overset{\circ}{W} = W$. Define the maps $\psi : X \to [0, 1],\ \psi(x) = (0 \vee \frac{p_W(x) - \varepsilon}{1 - \varepsilon}) \wedge 1, \varphi = \psi \circ u : T \to [0, 1]$ and $u_W = \varphi v \in (C_\Gamma(T) \otimes X)_{\text{loc}}$. For every $t \in T$, we have the equivalences:

$$\varphi(t) = 1 \Leftrightarrow p_W(u(t)) \geqslant 1 \Leftrightarrow u(t) \in X \setminus W \Leftrightarrow t \in T \setminus u^{-1}(W),$$

$$\varphi(t) = 0 \Leftrightarrow p_W(u(t)) \leqslant \varepsilon \Leftrightarrow u(t) \in \varepsilon\bar{W} \Leftrightarrow t \in u^{-1}(\varepsilon\bar{W}).$$

Since $v(T) \subset \operatorname{co}(u(T))$, $0 \in u(T)$, $0 \leqslant \varphi \leqslant 1$ and $u_W = \varphi v + (1-\varphi)0$, it follows that $u_W(T) \subset \operatorname{co}(u(T))$, $\operatorname{supp} u_W \subset \operatorname{supp} \varphi = \overline{T \backslash u^{-1}(\varepsilon \bar{W})} \subset T \backslash u^{-1}(\varepsilon W) = u^{-1}(X \backslash \varepsilon W) \subset u^{-1}(X \backslash \{0\})$. It remains to prove that $(u - u_W)(T) \subset W$. We have $u - u_W = u - \varphi v = (1-\varphi)u + \varphi(u - v)$. If $t \in T \backslash u^{-1}(W)$, then $(u - u_W)(t) = (u - v)(t) \in (u - v)(T) \subset W$. If $t \in u^{-1}(W)$, then $(u - u_W)(t) \in (1 - \varphi(t))W + \varphi(t)W = W$. Therefore, $(u - u_W)(T) \subset W$. ∎

Corollary 1. *If $T$ is quasi-compact, then for all $u \in C_X(T)$ and $W \in \mathscr{V}_X(0)$, there exists an approximant $u_W \in C_\Gamma(T) \otimes X$, such that*

$$(u - u_W)(T) \subset W, \ u_W(T) \subset \operatorname{co}(u(T)), \ \operatorname{supp} u_W \subset u^{-1}(X \backslash \{0\}).$$

Corollary 2. *$C_{\mathrm{tb}}(T, \Gamma) \otimes X$ is uniformly dense in $C_{\mathrm{tb}}(T, X)$.*

### *2.3. The case of $C_0(T, \Gamma) \otimes X \subset C_\infty(T, X)$*

Theorem 2. *Assume that $T$ and $X$ are Hausdorff. If $u \in C_\infty(T, X)$, then for all $W \in \mathscr{V}_X(0)$ and $K$ a compact subset of $T$, there exists an approximant $u_{W,K} \in C_0(T, \Gamma) \otimes X$, such that:*

*(1) $(u - u_{W,K})(T) \subset W$, $u_{W,K}(T) \subset \operatorname{co}(u(T) \cup \{0\})$, $u_{W,K}(K) \subset \operatorname{co}(u(T))$, $\operatorname{supp} u_{W,K} \subset u^{-1}(X \backslash \{0\})$,*

*(2) $u_{W,K} = \varphi \cdot \sum_{i \in I} \varphi_i(\cdot) x_i$, with $I$ finite, $(x_i)_{i \in I} \subset u(T)$, $(\varphi_i)_{i \in I}$ p.u. on $T$ and $\varphi : T \to [0, 1]$ a continuous map, such that $\varphi_{|_K} \equiv 1$.*

*Proof.* We can assume that $u \not\equiv 0$, that is $\exists t_0 \in T$, with $u(t_0) \neq 0$. Now fix $W \in \mathscr{V}_X(0)$, $W$ convex and $K \subset T$, $K$ compact. Set $M := K$ if $0 \notin u(K)$, and $M := \{t_0\}$ if $0 \in u(K)$. Since $u \in C_\infty(T, X) \subset C_{\mathrm{tb}}(T, X)$, by Theorem 1, $\exists v = \sum_{i \in I} \varphi_i(\cdot) x_i \in C_\Gamma(T) \otimes X$, such that $(u - v)(T) \subset W$, $v(T) \subset \operatorname{co}(u(T))$, $\operatorname{supp} v \subset u^{-1}(X \backslash \{0\})$, with $I$ finite, $(x_i)_{i \in I} \subset u(T)$ and $(\varphi_i)_{i \in I}$ p.u. on $T$. Since $0 \notin u(M)$ and $u(M)$ is compact, $\exists W_0 \in \mathscr{V}_X(0)$, such that $W_0 \subset W$, $W_0$ open and convex and $u(M) \cap W_0 = \emptyset$, that is $M \subset u^{-1}(X \backslash W_0)$. Now define $\psi : X \to [0, 1], \psi(x) = [0 \vee (2p_{W_0}(x) - 1)] \wedge 1$, $\varphi = \psi \circ u : T \to [0, 1]$, $w := \varphi v \in C_\Gamma(T) \otimes X$. For every $t \in T$, we have the following equivalences:

$$\varphi(t) = 1 \Leftrightarrow 2p_{W_0}(u(t)) \geqslant 2 \Leftrightarrow u(t) \notin W_0 \Leftrightarrow t \in u^{-1}(X \backslash W_0),$$

$$\varphi(t) = 0 \Leftrightarrow 2p_{W_0}(u(t)) \leqslant 1 \Leftrightarrow 2u(t) \in \bar{W}_0 \Leftrightarrow t \in u^{-1}(2^{-1} \bar{W}_0).$$

Clearly, $w(T) \subset [0, 1] \cdot v(T) \subset \operatorname{co}(u(T) \cup \{0\})$. Since $M \subset u^{-1}(X \backslash W_0)$, we have $\varphi_{|_M} \equiv 1$, and so $w(M) = v(M) \subset v(T) \subset \operatorname{co}(u(T))$ and $\operatorname{supp} \varphi = \overline{u^{-1}(X \backslash 2^{-1} \bar{W}_0)} \subset u^{-1}(X \backslash 2^{-1} W_0)$ compact. Hence, $w \in C_0(T, \Gamma) \otimes X$, $\operatorname{supp} w \subset \operatorname{supp} \varphi \subset u^{-1}(X \backslash \{0\})$. We next show that $(u - w)(T) \subset W$. If

$t \in u^{-1}(X \setminus W_0)$, then $(u-w)(t) = (u-v)(t) \in W$ and if $t \in u^{-1}(W_0)$, then $(u-w)(t) = (1-\varphi(t))u(t) + \varphi(t)(u-v)(t) \in (1-\varphi(t))W_0 + \varphi(t)W \subset W$. Hence, $(u-w)(T) \subset W$. If $M \neq K$, then $0 \in u(K)$, and so $w(K) \subset w(T) \subset \mathrm{co}(u(T) \cup \{0\}) = \mathrm{co}(u(T))$ and $w = 1 \cdot (\sum_{i \in I} (\varphi\varphi_i)x_i + (1-\varphi) \cdot 0)$. We conclude that $u_{W,K} := w$ satisfies all required properties. ∎

## 3. APPLICATIONS OF SPECIAL APPROXIMATIONS

### *3.1. The density of $C_0(T,\Gamma) \otimes X$ in $C_0(T,X)$ with respect to the inductive limit topology*

The following theorem is due to the constraint on the approximant's support.

THEOREM 3. *Assume that $T$ and $X$ are Hausdorff. If $u \in C_0(T,X)$, then for every $V \in \mathscr{V}_{C_0(T,X)}(0)$ with respect to the inductive limit topology, there exists an approximant $u_V \in C_0(T,\Gamma) \otimes X$, such that*:

(1) $u - u_V \in V$, $u_V(T) \subset \mathrm{co}(u(T))$, $\mathrm{supp}\, u_V \subset u^{-1}(X \setminus \{0\})$,

(2) $u_V = \sum_{i \in I} \varphi_i(\cdot)x_i$, *with $I$ finite, $(x_i)_{i \in I} \subset u(T)$ and $(\varphi_i)_{i \in I}$ p.u. on $T$.*

*Proof.* We can certainly assume that $0 \in u(T)$, since otherwise $T$ is compact and the conclusion is given by Theorem 1. Fix $V \in \mathscr{V}_{C_0(T,X)}(0)$ and set $K := \mathrm{supp}\, u$, $C_0(T,X)_K := \{w \in C_0(T,X) \mid \mathrm{supp}\, w \subset K\}$. Since $V \cap C_0(T,X)_K$ is a neighborhood of the origin in $C_0(T,X)_K$ with respect to the uniform convergence topology, $\exists W \in \mathscr{V}_X(0)$, such that $\{w \in C_0(T,X)_K \mid w(T) \subset W\} \subset V \cap C_0(T,X)_K$. Now Theorem 2 shows that $\exists v = \varphi \cdot \sum_{i \in I} \varphi_i(\cdot)x_i \in C_0(T,\Gamma) \otimes X$, with $I$ finite, $(x_i)_{i \in I} \subset u(T)$, $(\varphi_i)_{i \in I}$ p.u. on $T$, $\varphi : T \to [0,1]$ continuous, $\varphi_{|_K} \equiv 1$, and such that $(u-v)(T) \subset W$, $v(T) \subset \mathrm{co}(u(T))$, $\mathrm{supp}\, v \subset u^{-1}(X \setminus \{0\}) \subset K$. We thus get $u - v \in C_0(T,X)_K$, $(u-v)(T) \subset W$, and so $u - v \in V$. Since $v = \sum_{i \in I} (\varphi\varphi_i)x_i + (1-\varphi) \cdot 0$, $u_V := v$ satisfies all required properties. ∎

COROLLARY 3. *If $T, X$ are Hausdorff, then $C_0(T,\Gamma) \otimes X$ is dense in $C_0(T,X)$ with respect to the inductive limit topology. Moreover, if $X$ is metrizable, then this density is sequential.*

*Proof.* The proof is immediate, with Theorem 3. ∎

### *3.2. A Tietze–Dugundji's type extension theorem*

In this subsection, $T$ denotes a topological space and $X$ a $\Gamma$-locally convex Hausdorff space. The following two lemmas emphasize the existing connection between approximation and extension theorems.

LEMMA 1. *Assume that $X$ is also a Fréchet space. Then, for every subset $F \subset T$, the following two statements are equivalent*:

(1) *For all* $u \in C_X(F)$ and $W \in \mathscr{V}_X(0)$, *there exists* $u_W \in C_X(T)$, *such that* $(u - u_W)(F) \subset W$.

(2) *For every* $u \in C_X(F)$, *there exists* $\tilde{u} \in C_X(T)$, such that $\tilde{u}_{|_F} = u$ and $\tilde{u}(T) \subset \operatorname{co}(\overline{u(F)})$.

*Proof.* It is to prove (1) $\Rightarrow$ (2). Let us first show that $\forall u \in C_X(F)$, $\forall W \in \mathscr{V}_X(0), \exists v \in C_X(T)$, with $(u - v)(F) \subset W, v(T) \subset \operatorname{co}(u(F)) + W$. Fix $u \in C_X(F)$, $W \in \mathscr{V}_X(0)$ and choose $W_0 \in \mathscr{V}_X(0)$, $W_0$ balanced, convex, such that $2W_0 \subset W$. According to our hypothesis, $\exists w \in C_X(T)$, with $(u - w)(F) \subset W_0$. It follows that $\overline{w(F)} \subset w(F) + W_0 \subset \operatorname{co}(u(F)) + 2W_0$, and so $\operatorname{co}(\overline{w(F)}) \subset \operatorname{co}(u(F)) + 2W_0 \subset \operatorname{co}(u(F)) + W$. Now define the map $f : \overline{w(F)} \to X$, $f(x) = x$. By Dugundji's theorem, $\exists \tilde{f} \in C_X(X)$, such that $\tilde{f}_{|_{\overline{w(F)}}} = f$ and $\tilde{f}(X) \subset \operatorname{co}(\overline{w(F)})$. Set $v := \tilde{f} \circ w \in C_X(T)$. It follows that $v_{|_F} = \tilde{f} \circ w_{|_F} = w_{|_F}$, $(u - v)(F) = (u - w)(F) \subset W_0 \subset W$, $v(T) \subset \tilde{f}(X) \subset \operatorname{co}(\overline{w(F)}) \subset \operatorname{co}(u(F)) + W$. To finally prove (2), fix again $u \in C_X(F)$. Since $X$ is metrizable, we can choose $(W_n)_{n \in \mathbf{N}}$ a fundamental system of convex neighborhoods of the origin in $X$, with $2W_{n+1} \subset W_n\ \forall n \in \mathbf{N}$. For $u_0 := u \in C_X(F)$, $\exists v_0 \in C_X(T)$, such that $(u_0 - v_0)(F) \subset W_0$ and $v_0(T) \subset \operatorname{co}(u_0(F)) + W_0$. Set $u_1 := u_0 - v_0|_F \in C_X(F)$. Thus, we can inductively define $(u_n)_{n \in \mathbf{N}} \subset C_X(F)$, $(v_n)_{n \in \mathbf{N}} \subset C_X(T)$, such that $\forall n \in \mathbf{N}$, we have: $(u_n - v_n)(F) \subset W_n$, $v_n(T) \subset \operatorname{co}(u_n(F)) + W_n$, $u_{n+1} := u_n - v_n|_F$. Hence, $\forall n \in \mathbf{N}, u_{n+1}(F) = (u_n - v_n)(F) \subset W_n$, $v_{n+2}(T) \subset \operatorname{co}(u_{n+2}(F)) + W_{n+2} \subset W_{n+1} + W_{n+2} \subset 2W_{n+1} \subset W_n$. Therefore, $u_n \xrightarrow{\text{u.}} 0$ on $F$ and $\sum_{n \geqslant 0} v_n$ is uniformly convergent on $T$. Set $v := \sum_{n=0}^{\infty} v_n \in C_X(T)$. But $\forall t \in F$, we have $v(t) = \lim_{n \to \infty} \sum_{j=0}^{n} v_j(t) = \lim_{n \to \infty} \sum_{j=0}^{n} (u_j(t) - u_{j+1}(t)) = u(t)$, and so $v_{|_F} = u$. As before, $\exists \tilde{g} \in C_X(X)$, with $\tilde{g}(x) = x\ \forall x \in \overline{u(F)}$ and $\tilde{g}(X) \subset \operatorname{co}(\overline{u(F)})$. For $\tilde{u} := \tilde{g} \circ v \in C_X(T)$, we clearly have $\tilde{u}_{|_F} = u$ and $\tilde{u}(T) \subset \operatorname{co}(\overline{u(F)})$. ∎

LEMMA 2. *Assume that $X$ is also a Fréchet space. Then, for every subset $F \subset T$, the following two statements are equivalent*:

(1) *For all $u \in C_{\text{tb}}(F, X)$ and $W \in \mathscr{V}_X(0)$, there exists $u_W \in C_{\text{tb}}(T, X)$, such that $(u - u_W)(F) \subset W$.*

(2) *For every $u \in C_{\text{tb}}(F, X)$, there exists $\tilde{u} \in C_{\text{tb}}(T, X)$, such that $\tilde{u}_{|_F} = u$ and $\tilde{u}(T) \subset \operatorname{co}(\overline{u(F)})$.*

*Proof.* It is to show (1) $\Rightarrow$ (2). The proof is similar to that of Lemma 1, observing first that $\forall u \in C_{\text{tb}}(F, X), \forall W \in \mathscr{V}_X(0), \exists v \in C_{\text{tb}}(T, X)$, with $(u - v)(F) \subset W$, $v(T) \subset \operatorname{co}(u(F)) + W$. Consequently, $u - v \in C_{\text{tb}}(F, X)$. Thus, the same construction as in the previous proof finally leads to the maps $v := \sum_{n=0}^{\infty} v_n \in C_{\text{tb}}(T, X)$ and $\tilde{u} := \tilde{g} \circ v \in C_{\text{tb}}(T, X)$. ∎

The following lemma is a variant of Theorem 1.

LEMMA 3. *Assume that* $T$ *is normal. Consider a closed subset* $F \subset T$ *and* $u \in C_X(F)$. *If* $T$ *is paracompact or* $u \in C_{\mathrm{tb}}(F, X)$, *then for every* $W \in \mathscr{V}_X(0)$, *there is a* $u_W \in (C_\Gamma(T) \otimes X)_{\mathrm{loc}}$, *with* $(u - u_W)(F) \subset W$ *and* $u_W(T) \subset \mathrm{co}(u(F))$. *Moreover, if* $u \in C_{\mathrm{tb}}(F, X)$, *then we can find* $u_W \in C_\Gamma(T) \otimes X$.

*Proof.* Fix $W \in \mathscr{V}_X(0)$, with $W$ open and convex. If $u \in C_{\mathrm{tb}}(F, X)$, then $\exists A_0 \subset u(F)$, with $A_0$ finite and $u(F) \subset A_0 + W$. Set $A := A_0$ if $u \in C_{\mathrm{tb}}(F, X)$, and $A := u(F)$, otherwise. Therefore, $u(F) \subset A + W$, $F = u^{-1}(A + W) = \bigcup_{x \in A} u^{-1}(x + W)$. For every $x \in A$, $u^{-1}(x + W)$ is open in $F$, and so $\exists U_x$ open in $T$, with $u^{-1}(x + W) = U_x \cap F$. Thus, $T = (T \backslash F) \cup \bigcup_{x \in A} U_x = \bigcup_{x \in B} U_x$, where $B := A \cup \{A\}$, $U_A := T \backslash F$. If $u \in C_{\mathrm{tb}}(F, X)$, then $(U_x)_{x \in B}$ is a finite open covering of $T$, which is a normal space. If $u \notin C_{\mathrm{tb}}(F, X)$, then $(U_x)_{x \in B}$ is an open covering of the paracompact space $T$. In both cases, $\exists (\varphi_x)_{x \in B}$ p.u. on $T$, subordinated to $(U_x)_{x \in B}$. Hence, $\varphi_A = 1 - \sum_{x \in A} \varphi_x$, $\mathrm{supp}\, \varphi_A \subset T \backslash F$. Now choose $z \in u(F)$ and consider $u_W := \sum_{x \in A} \varphi_x(\cdot) x + \varphi_A(\cdot) z \in (C_\Gamma(T) \otimes X)_{\mathrm{loc}}$. Obviously, $u_W(T) \subset \mathrm{co}(u(F))$. We next show that $(u - u_W)(F) \subset W$. Fix $t \in F$ and set $J := \{x \in A \mid \varphi_x(t) \neq 0\}$. Thus, $\sum_{x \in J} \varphi_x(t) = 1$, $(u - u_W)(t) = \sum_{x \in J} \varphi_x(t) \times (u(t) - x)$. But $\forall x \in J$ we have $\varphi_x(t) \neq 0$, and so $t \in \mathrm{supp}\, \varphi_x \subset U_x$, $u(t) \in x + W$. We thus get $(u - u_W)(t) \in \sum_{x \in J} \varphi_x(t) W = W$. Hence, $u_W$ satisfies all required properties. ∎

THEOREM 4. *Assume that* $T$ *is normal and* $X$ *is Fréchet. Consider a closed subset* $F \subset T$ *and* $u \in C_X(F)$. *If* $T$ *is paracompact or* $u \in C_{\mathrm{tb}}(F, X)$, *then there exists* $\tilde{u} \in C_X(T)$, *such that* $\tilde{u}_{|_F} = u$ *and* $\tilde{u}(T) \subset \mathrm{co}(\overline{u(F)})$.

*Proof.* By Lemma 3, $F$ satisfies condition (1) of Lemma 1 or 2. In both cases, $\exists \tilde{u} \in C_X(T)$, such that $\tilde{u}_{|_F} = u$ and $\tilde{u}(T) \subset \mathrm{co}(\overline{u(F)})$. ∎

The above theorem is a strengthening of Theorem 3.6 [6], since $u$ need not be a compact map (if $T$ is paracompact) and $X$ need not be a Banach space. For a metrizable space $T$, we recover Dugundji's extension theorem (see [2, 3]). The following corollary is known (see [6, Corollary 3.5, p. 54]).

COROLLARY 4. *Assume that* $T$ *is completely regular and that* $X$ *is metrizable. Consider a compact subset* $F \subset T$ *and* $u \in C_X(F)$. *Then, there exists* $\tilde{u} \in C_X(T)$, *such that* $\tilde{u}_{|_F} = u$ *and* $\tilde{u}(T) \subset \mathrm{co}(u(F))$.

*Proof.* Since $T$ is completely regular, $\exists \tilde{T}$ a compact topological space, such that $T$ is a dense subspace of $\tilde{T}$. Let $\tilde{X}$ denote the completion of $X$. Since $u \in C_{\tilde{X}}(F)$ and $F$ is compact in $\tilde{T}$, Theorem 4 shows that $\exists v \in C_{\tilde{X}}(\tilde{T})$, such that $v_{|_F} = u,\ v(\tilde{T}) \subset \overline{\mathrm{co}(u(F))} = \mathrm{co}(u(F)) \subset X$. We thus get $\tilde{u} := v_{|_T} \in C_X(T)$, $\tilde{u}_{|_F} = u,\ \tilde{u}(T) \subset \mathrm{co}(u(F))$. ∎

COROLLARY 5. *Assume that $T$ is $\sigma$-compact and that $X$ is metrizable. Consider a closed subset $F \subset T$ and $u \in C_X(F)$. Then, there exists $\tilde{u} \in C_X(T)$, such that $\tilde{u}_{|_F} = u$ and $\tilde{u}(T) \subset \mathrm{co}(u(F))$.*

*Proof.* By hypothesis, $\exists (K_n)_{n\in\mathbf{N}}$ a family of compact subsets of $T$, such that $T = \bigcup_{n\in\mathbf{N}} K_n$ and $K_n \subset \overset{\circ}{K}_{n+1}$ $\forall n \in \mathbf{N}$. Set $F_n := F \cap K_n$ $\forall n \in \mathbf{N}$. Since $F_0$ is compact, $K_0$ is normal and $u_{|_{F_0}} \in C_X(F_0)$, by Corollary 4, $\exists u_0 \in C_X(K_0)$, with $u_0|_{F_0} = u_{|_{F_0}},\ u_0(K_0) \subset \mathrm{co}(u(F_0))$. For fixed $n \in \mathbf{N}$, assume that $\exists u_n \in C_X(K_n)$, such that $u_n|_{F_n} = u_{|_{F_n}}, u_n(K_n) \subset \mathrm{co}(u(F_n))$. Define $v \in C_X(K_n \cup F_{n+1})$ by $v_{|_{K_n}} = u_n,\ v_{|_{F_{n+1}}} = u_{|_{F_{n+1}}}$. By Corollary 4, it follows that $\exists u_{n+1} \in C_X(K_{n+1})$, such that $u_{n+1}|_{(K_n\cup F_{n+1})} = v$ and $u_{n+1}(K_{n+1}) \subset \mathrm{co}(v(K_n \cup F_{n+1}))$. This easily gives $u_{n+1}|_{F_{n+1}} = u_{|_{F_{n+1}}},\ u_{n+1}(K_{n+1}) \subset \mathrm{co}(u_n(K_n) \cup u(F_{n+1})) = \mathrm{co}(u(F_{n+1}))$ and $u_{n+1}|_{K_n} = u_n$. Therefore, we can inductively define $u_n \in C_X(K_n)$, such that $\forall n \in \mathbf{N}$, we have $u_{n+1}|_{K_n} = u_n$, $u_n|_{F_n} = u_{|_{F_n}}$ and $u_n(K_n) \subset \mathrm{co}(u(F_n))$. It follows that $\exists \tilde{u} \in C_X(T)$, defined by $\tilde{u}_{|_{K_n}} = u_n$ $\forall n \in \mathbf{N}$. Obviously, $\tilde{u}_{|_F} = u$ and $\tilde{u}(T) \subset \mathrm{co}(u(F))$. ∎

### *3.3. A proof of Schauder–Tihonov's fixed point theorem*

LEMMA 4. *Let $T$ be a topological space, $Y$ a $\Gamma$-topological vector space and $(u_\delta)_{\delta\in\Delta} \subset C_Y(T), (t_\delta)_{\delta\in\Delta} \subset T$* nets, such that $t_\delta \to t \in T$ and $u_\delta \overset{\mathrm{u.}}{\to} u$, $u : T \to Y$. Then, $u_\delta(t_\delta) \to u(t)$.

*Proof.* The proof is standard. ∎

The following proof is due to the constraint on the approximant's range.

THEOREM 5 (Schauder–Tihonov). *Let $X$ be a $\Gamma$-locally convex Hausdorff space, $M$ a closed convex subset of $X$ and $U : M \to M$ a completely continuous operator. Then, $U$ has a fixed point.*

*Proof.* Since $U$ is completely continuous, $K := \overline{U(M)}$ is compact in $X$. Clearly, $K \subset M$. On $\mathscr{W} := \{W \in \mathscr{V}_X(0) \mid W$ is balanced$\}$, we consider the usual order relation, given by: $W_1 \preceq W_2 \Leftrightarrow W_1 \supset W_2$. Now fix $W \in \mathscr{W}$. Since $U \in C_{\mathrm{tb}}(M, X)$, by Theorem 1, $\exists U_W = \sum_{i\in I} \varphi_i(\cdot)x_i \in C_\Gamma(M) \otimes X$, such that $(U - U_W)(M) \subset W$, $(x_i)_{i\in I} \subset U(M), (\varphi_i)_{i\in I}$ p.u. on $M$ for some finite set $I$. Set $K_W := \mathrm{co}(\{x_i \mid i \in I\}) \subset \mathrm{co}(U(M)) \subset M,\ X_W := \mathrm{Sp}(K_W)$. But $X_W$ is clearly normable since it is Hausdorff and has finite dimension, $K_W$ is convex and compact and $U_W(K_W) \subset U_W(M) \subset K_W$. Now Brouwer's fixed point theorem shows that $\exists x_W \in K_W$, such that $U_W(x_W) = x_W$. This leads to $x_W = U_W(x_W) \in U_W(M) \subset U(M) + W \subset K + W$, and so $\exists y_W \in K$, such that $x_W - y_W \in W$. We thus get $(U_W)_{W \in \mathscr{W}} \subset C_X(M)$, $(x_W)_{W \in \mathscr{W}} \subset M$ and $(y_W)_{W \in \mathscr{W}} \subset K$, nets with the above properties. But $(U - U_W)(M) \subset W$ and $x_W - y_W \in W \ \forall W \in \mathscr{W}$, lead to $U_W \xrightarrow{\text{u.}} U$ and $x_W - y_W \to 0$. As $K$ is compact, $(y_W)_{W \in \mathscr{W}} \subset K$ has a subnet $(y_{\varphi(\delta)})_{\delta \in \Delta}$, convergent to an element $\xi \in K$. Therefore, $U_{\varphi(\delta)} \xrightarrow{\text{u.}} U$ and $x_{\varphi(\delta)} \to \xi$. Now Lemma 4 gives $x_{\varphi(\delta)} = U_{\varphi(\delta)}(x_{\varphi(\delta)}) \to U(\xi)$, and so $U(\xi) = \xi$. ∎

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